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Relaxation Methods Applied to Engineering Problems. V. Conformal Transformation of a Region in Plane Space

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RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

V. CONFORMAL TRANSFORMATION OF A REGION IN PLANE SPACE

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Introduction

1. Part III of this series (Christopherson and Southwell 1938) brought within the scope of relaxation methods problems governed by Poisson's or Laplace's equation in two variables. Its first example (torsion of an isotropic cylinder) was a standard problem in potential theory, calling for the evaluation within a given region of a plane-harmonic function defined as having specified values at the boundary. This was solved without difficulty, and with more than sufficient accuracy for practical purposes.

Conformal transformation, the problem considered here, although essentially similar presents some questions of detail which were not encountered in Part III. Regarded as a weapon of the analyst, it is a means whereby problems relating to specified regions in plane space may be reduced to problems which concern regions of simpler shape (e.g. circles or rectangles) and can be solved in terms of known functions of ordinary (e.g. polar or Cartesian) co-ordinates. Contours of these co-ordinates by intersection divide the simpler region into rectangles: conformal transformation entails a similar division, or "mapping", of the specified region by intersecting contours of two conjugate plane-harmonic functions, α, β, which in turn serve as co-ordinates to define the position of any point. Orthodox mathematics presents the transformation in an equation of the type

$$\alpha+i\beta=f(x+iy), \tag{1}$$

which, when the form of f is known, expresses a one-to-one relation between points in the first region and in the second; but this functional relation is of no concern to the practical computer provided that he can construct "maps" of which it is the mathematical expression, and for this it is only necessary to have α and β evaluated at nodal points of some regular "net". The procedure whereby contours are constructed is so obvious as not to require description.

Ordinarily one of the two functions (β, say) must take specified values at the boundary, therefore can be evaluated elsewhere by the methods of Part III. But values so calculated will not constitute an exact solution of Laplace's equation, and as such will not be compatible with the existence of a single-valued function α: therefore we have still to devise a method (not required in any problem treated hitherto) whereby having approximated to a plane-harmonic function we may deduce a similar approximation to its conjugate. Again, we shall sometimes want to transform a region which has infinite extent (e.g. the region *external* to a specified boundary): then special devices will be needed, since Part III was concerned only with restricted fields. In some applications,* having approximated to α or β, we shall require a corresponding approximation to the quantity $h = |\, d(\alpha+i\beta)/d(x+iy)\,|$, defined by

$$h^2 = \left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 = \left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2. \tag{2}$$

In the present paper these requisite extensions of technique, together with certain minor modifications, are described in relation to specific examples. For brevity the descriptions terminate with the approximate calculation of α and β, but the contour "maps" are reproduced. It should be remarked that in general the accuracy of the calculations is greater than a map can exhibit.

2. Clearly the transformed region must be such that the governing equation (which may of course be altered as a consequence of the transformation) can be solved in terms of known functions. The following are commonly employed:

(*a*) transformation of the region enclosed within a given boundary into the region enclosed within a circle;

(*b*) transformation of the region enclosed within a given boundary into the region enclosed within a rectangle;

(*c*) transformation of the region external to a given boundary into the region external to a circle;

(*d*) transformation of a semi-infinite region into a half-plane or infinite strip.

Figure 1 shows the nature of these transformations and the form of (1) which is appropriate to each.

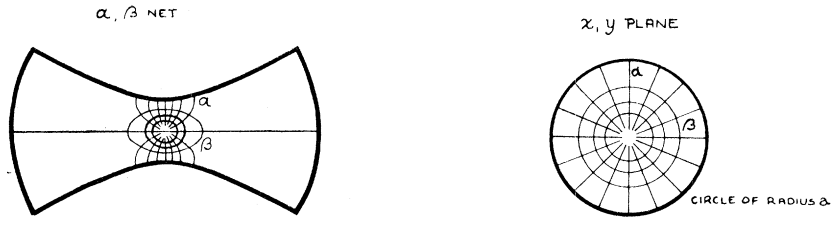
In illustration of (*a*) we have transformed the region contained by an ellipse and the two branches of a confocal hyperbola. The centres of the ellipse and circle are corresponding points, and the "nets" are formed by closed curves surrounding these points and by lines radiating from them. Such nets could be used (although it is not the most direct method)† to solve the torsion problem of Saint Venant, and the section has some interest from this standpoint as being one of the examples chosen by Filon (1899) to illustrate his contention that the shear stress is not always (as stated by Boussinesq) greatest at those points of the boundary which are nearest the centroid.

In illustration of (*b*) we have transformed a typical rail section. The resulting map could be employed as a means of solving the flexure problem for this section, and in this connexion would have advantages as compared with the map resulting from a transformation of type (*a*).

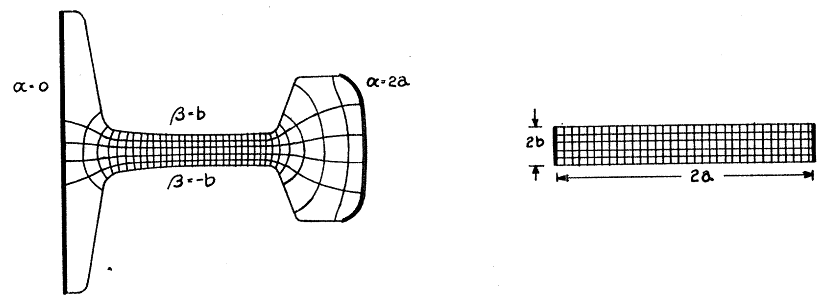
* E.g. in Hydrodynamics, where h measures the resultant velocity corresponding with a stream-function β.

† Its application to the equilateral-triangular section was described in a recent paper (Bradfield, Hooker and Southwell 1937, §§ 20–26).

In illustration of (*c*) we have transformed a typical airscrew section which was studied from an aerodynamical standpoint by Bryant and Williams (1925), using the device of geometrical "inversion" to reduce the problem (as regards the application of relaxation methods) to a further example of the transformation (*a*).

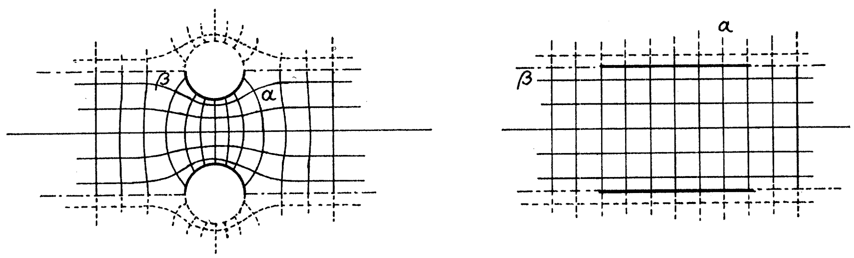


(*a*)



(*b*)

(*c*)



(*d*)

FIGURE 1

(*a*) $x+iy = ae^{i(\alpha+i\beta)}$
(*b*) $x+iy = \alpha+i\beta$
(*c*) $x+iy = ae^{-i(\alpha+i\beta)}$
(*d*) $x+iy = \alpha+i\beta$

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In illustration of (d) we have transformed an infinite plane containing a row of circular holes, equally spaced, into an infinite plane divided into parallel strips.

3. Our last problem (section V) is in essence a further example of conformal transformation, but electrical rather than mathematical aspects are stressed in the discussion.

An electrical interpretation of conformal transformation was propounded in a paper (Bradfield, Hooker and Southwell 1937) written before the application of relaxation methods to this purpose. It was shown (§ 12) that in every instance two boundaries are really involved, although these usually reduce to a closed curve and an interior point. The function β can be interpreted as electric potential in the (two-dimensional) field between the boundaries, these being visualized as conductors to which a steady potential-difference is applied: so an electrical measurement will serve to determine β, and conversely, if in theory we can perform the operation of conformal transformation for any specified boundaries, then we can calculate the electric capacity of any straight cable or condenser. It will not usually be necessary to determine contours of the conjugate function α, but these lines (which give the direction of the electric field) can be plotted if required.

Multi-core cables have important uses in electrical engineering, and they are commonly enclosed for protection in a metallic sheath. To calculate the capacitance of a given arrangement may be very difficult if orthodox methods are employed, and approximate methods will be correspondingly valuable. The use for this purpose of relaxation methods is exemplified in §§ 18–20.

We acknowledge with gratitude help received from the Secretary and Staff of the Aeronautical Research Committee in the preparation of diagrams suitable for reproduction.

I. Transformation of an interior region into a circle

The problem

4. This (cf. § 3) is a special case of the more general transformation whereby a region contained between two closed boundaries A and B is transformed into the region contained between two concentric circles. Of the two plane-harmonic functions α, β (§ 1), β must have a constant value on each of A, B, and α (its conjugate) will be cyclic.

Without loss of generality β may be required to vanish on the outer boundary A and to have the value 1 at every point of the inner boundary B. Then β will be definite, also α and its cyclic constant $\bar{\alpha}$. Writing

$$\gamma = \alpha + i\beta, \quad z = x + iy, \tag{3}$$

where x and y stand for Cartesian co-ordinates in the transformed region, we can express the transformation by

$$\frac{z}{a} = \left(\frac{a}{b}\right)^{i\gamma}, \tag{4}$$

as is easily verified; for according to (4)

$$\left.\begin{aligned} \frac{z}{a} &= \left(\frac{b}{a}\right)^{\beta} (\cos\theta + i\sin\theta), \\ \text{where} \quad \theta &= \alpha \log\frac{a}{b}, \end{aligned}\right\} \tag{5}$$

and thus the contours $\beta = 0$, $\beta = 1$ transform into concentric circles of radius a, b.

In order that the region between A and B may transform into a *complete* circular annulus, θ must be cyclic with constant 2π. Therefore according to (5) the relation

$$\log\frac{a}{b} = 2\pi/\bar{\alpha}$$

must be satisfied, so that while *either* a or b may be chosen at will, the ratio a/b must have a particular value. In the paper cited (§ 3) this result was given a physical interpretation: the original and the transformed region, regarded as conducting sheets of equal resistivity, must offer the same total resistance to the passage of electric current between their inner and outer boundaries. But on that understanding, if (as in the present example) B is contracted indefinitely, then a/b will tend to infinity: therefore *any* closed boundary A, and *any* interior point B, can be transformed into a complete circle and its centre.

5. Now, by reason of the singularity at B, it is convenient to express the solution in two parts and write

$$\alpha = \theta - \phi, \quad \beta = \log\frac{r_0}{r} - \psi, \tag{6}$$

r, θ being polar co-ordinates *in the specified region* and r_0 being arbitrary. Then ϕ and ψ, like α and β, will be conjugate plane-harmonic functions, and having no singularity inside A they will be defined completely (except as regards a nugatory constant in α) by the requirement that $\beta = 0$ at every point on that boundary. Giving γ and z the same significance as in (3), but now replacing (4) by the transformation

$$\begin{aligned} \frac{z}{a} = e^{i\gamma} &= e^{-\beta}(\cos\alpha + i\sin\alpha) \\ &= \frac{r}{r_0} e^{\psi} \{\cos(\theta-\phi) + i\sin(\theta-\phi)\}, \end{aligned} \tag{7}$$

we see that this will transform A (i.e. the contour $\beta = 0$) into a circle of chosen radius a, and B (where $r = 0$ and ψ is finite) into the origin $z = 0$. Since ϕ is single-valued, α like θ will be cyclic with cyclic constant 2π.

Calculation by relaxation methods of the "potential function" ψ

6. We proceed to calculate ψ (and therefore β) and from it to deduce ϕ (and therefore α) for a doubly-symmetrical region bounded by an ellipse and the two branches of a confocal hyperbola (Filon's torsion section: cf. § 2). As the point B we shall take the

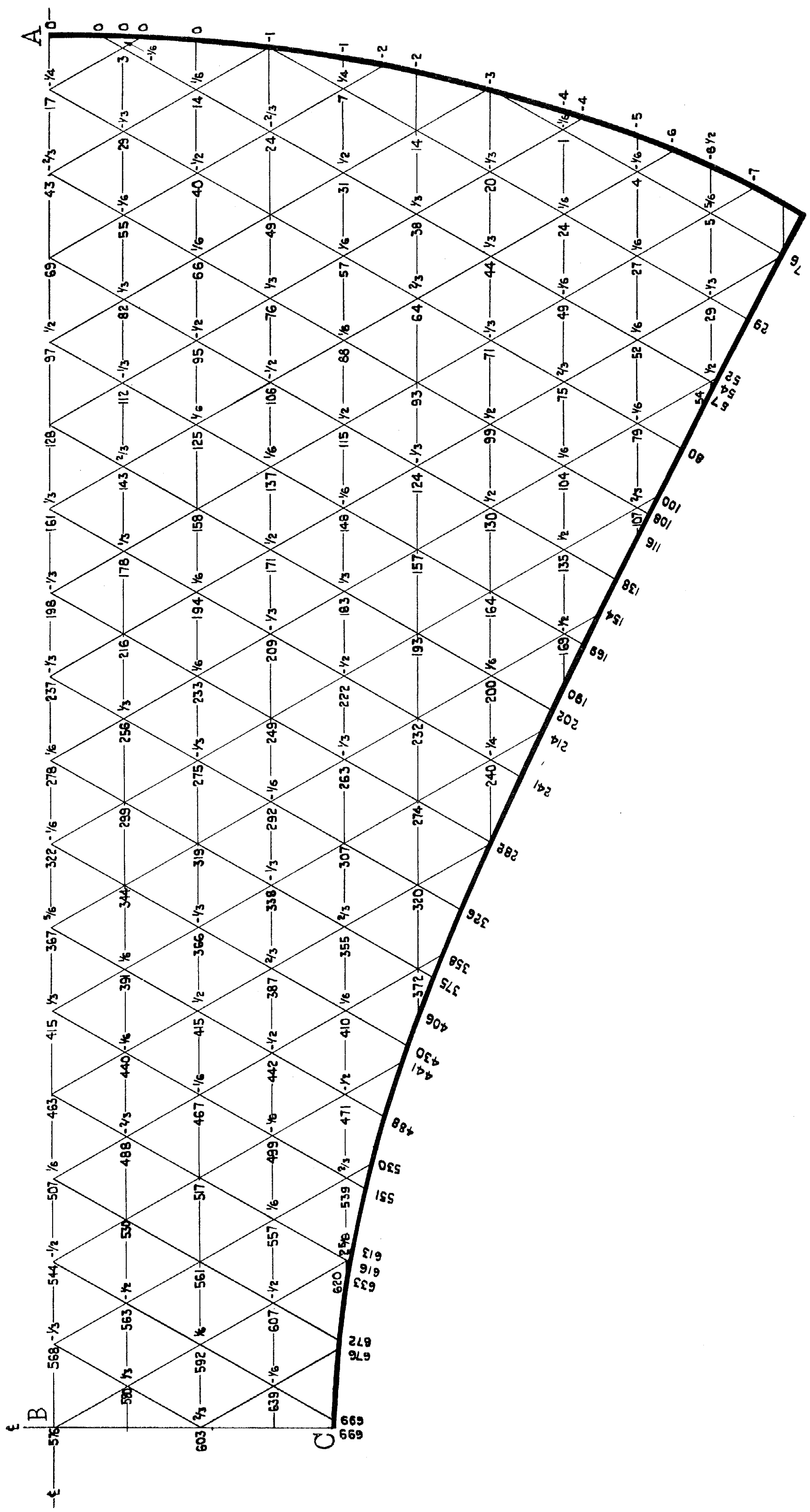

FIGURE 2

common centre of the ellipse and hyperbola. We term ψ the "potential function", ϕ the "stream function" of the modified problem.

It is not necessary to give details of the approximate determination of ψ, because an exactly similar problem has been discussed in Part III of this series (Christopherson and Southwell 1938). Triangular nets were employed, with meshes successively reduced in accordance with Part III, § 13. In figure 2, which shows the finest net of the series, figures immediately to the left of nodal points give ψ as estimated from the finite-difference relation, figures to the right give "residual forces", in the manner of Part III, § 16.* As usual all figures have been multiplied with the aim of avoiding decimals in computation. If r_0 (§ 5) is identified with the distance BA in figure 2, the boundary values there given are such as satisfy the equation

$$\psi + 1000 \log_{10}(r/r_0) = 0, \tag{8}$$

and comparison of (8) with (6) shows that in consequence the calculated values (at internal points) contain a multiplying factor

$$1000 \log_{10} e = 434{\cdot}2945.$$

Determination of the conjugate "stream function" ϕ

7. Having evaluated ψ with an accuracy deemed to be sufficient, to complete the solution we must associate with it a similar approximation to the conjugate harmonic ϕ. The exact solutions for ϕ and ψ satisfy the relations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}, \tag{9}$$

and are, moreover, without singularities and single-valued in the sense that both $\oint \frac{\partial \phi}{\partial s} ds$ and $\oint \frac{\partial \psi}{\partial s} ds$ vanish when the integrals relate to any closed curve which lies within the outer boundary A. But our solution for ψ, being only an approximation, though single-valued is not strictly plane-harmonic: therefore it is not possible to satisfy both of (9) by any single-valued approximation to ϕ.

8. The two conditions (9) require that the first shall hold when Ox, Oy are rotated through any angle. Accordingly in figure 3, $\partial\phi/\partial x'$ should be equal to $\partial\psi/\partial y'$, and by expanding both sides according to Taylor's series it is easy to show that the relation

$$\phi_Q - \phi_P = \frac{a}{b}(\psi_R - \psi_S) \tag{10}$$

is equivalent, if in the series terms of the third and higher orders in a and b are neglected in comparison with a or b. Similar approximations being the essence of the relaxation treatment, we may consistently replace (9), in what follows, by the finite-difference relation (10).

* On account of symmetry only one quarter of the whole net requires to be reproduced.

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Suppose that ψ has been determined (approximately) at nodal points of the triangular net shown by fine lines in figure 4. In that diagram the points lettered P, Q, R, S have the same spatial relation as in figure 3 if

$$\frac{a}{b} = \frac{PQ}{RS} = \frac{1}{\surd 3}:$$

therefore according to (10) we have

$$\phi_Q - \phi_P = \frac{\psi_R - \psi_S}{\surd 3}, \tag{11}$$

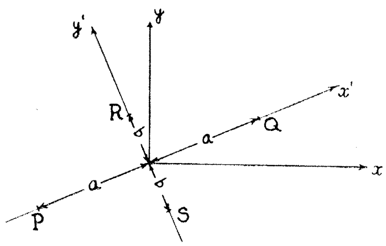

FIGURE 3

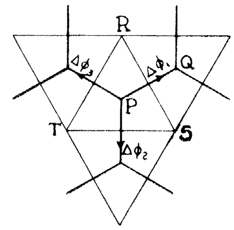

FIGURE 4

and so, knowing ψ, we can attach a numerical value to $\Delta\phi_1$. (An arrow is used to indicate the direction in which this increment is measured.) In exactly the same way we can attach a unique value to $\Delta\phi$ for every link of the bold-line hexagonal net in figure 4.

Because ψ is single-valued the circuital relation $\oint \frac{\partial \psi}{\partial s}\, ds = 0$ is satisfied for the triangle RST, and hence

$$\Delta\phi_1 + \Delta\phi_2 + \Delta\phi_3 = \Sigma_3(\phi) - 3\phi_P = 0. \tag{12}$$

This means that in the immediate neighbourhood of any nodal point of the hexagonal net ϕ as calculated from the $\Delta\phi$'s will satisfy exactly (for $N = 3$) the finite-difference approximation to Laplace's equation (cf. Part III, § 8). But because ψ does not exactly satisfy the similar relation (with $N = 6$) appropriate to its triangular net, the integral $\oint \frac{\partial \phi}{\partial s}\, ds$ taken round one of the bold-line hexagons will not in general vanish. This means that the $\Delta\phi$'s will *not* in general determine a single-valued function ϕ.

9. Starting with ψ as presented (e.g.) in figure 2, we can deduce $\Delta\phi$'s in accordance with (11) and from these, having given an arbitrary value to ϕ at any one point, deduce its value at any other by summing along any selected path. Values so calculated will be unique provided that the path is not self-cutting; but they will *not* constitute a unique solution, because different values would have been obtained if another path had been chosen. To that extent they are indeterminate, and we have now to eliminate this undesirable quality. In other words, only some of the calculated $\Delta\phi$'s have been used to obtain a solution, and the same weight should be given to all.

When ϕ is interpreted mechanically in the manner of Part III, § 9 (i.e. as transverse displacement of an actual net having the same tension in every link) a procedure for adjusting calculated $\Delta\phi$'s is at once apparent. When the values of ϕ at two adjacent nodes are not compatible with the value which has been attached to $\Delta\phi$ for the link connecting them, then before this link can be fitted into place its ends must be given a relative displacement $\Delta\phi'$ additional to what it has in virtue of $\Delta\phi$: therefore according to the convention of Part III (§ 18) transverse forces $\pm\Delta\phi'/N$ must be applied to its ends, involving equal "residual forces" on the corresponding "constraints". Eliminating in this manner every discrepancy of the type denoted by $\Delta\phi'$, we obtain a solution for ϕ which is single-valued but entails residual forces: these must be liquidated in order that it may satisfy our approximation (12) to Laplace's equation.

Alternatively the total residual force at any point can be calculated from the formula

$$\bar{\bar{F}} = \frac{1}{N}\Sigma_{a,N}(\phi) - \phi_0 \tag{13}$$

of Part III, § 11. The result will be the same, since the right-hand side of (13)

$$= \frac{1}{N}\Sigma_N(\Delta\phi + \Delta\phi'), \text{ in the notation used above,}$$

$$= \frac{1}{N}\Sigma(\Delta\phi'),$$

because the $\Delta\phi$'s will sum to zero as exemplified in (12).

10. There are now two possibilities: either the residual forces are of an order neglected in the ψ-solution, and so with consistency can be disregarded, or they will call for liquidation by the methods of Part III. In the latter event (since we have seen that every discrepancy entails equal and opposite forces at adjacent nodes) liquidation may be expected to be rapid.

Greater accuracy can be attained by dealing with ϕ on the basis of a triangular instead of a hexagonal net, and the change can be made without difficulty either before or after the final liquidation. For each hexagon centre we can calculate ϕ from the finite-difference equation

$$\phi_0 = \frac{1}{N}\Sigma_{a,N}(\phi), \tag{14}$$

with $N = 6$: then the residual force will vanish at the hexagon centres, and for a corner it can be calculated in terms of the six surrounding values of ϕ from the formula

$$\bar{\bar{F}} = \frac{1}{N}\Sigma_{a,N}(\phi) - \phi_0, \tag{13) bis$$

N again having the value 6.

11. Figure 5 serves to illustrate the suggested procedure. In its left-hand portion values of ψ (from figure 2) are shown for the hexagon centres, and numbers are attached to $\Delta\phi$ for every hexagon side.* In its central portion are given values of ϕ deduced from

* The factor $1/\sqrt{3}$ which occurs in equation (11) has here been disregarded, but the given values of ϕ were divided by $\sqrt{3}$ before the α-contours (figure 6) were constructed.

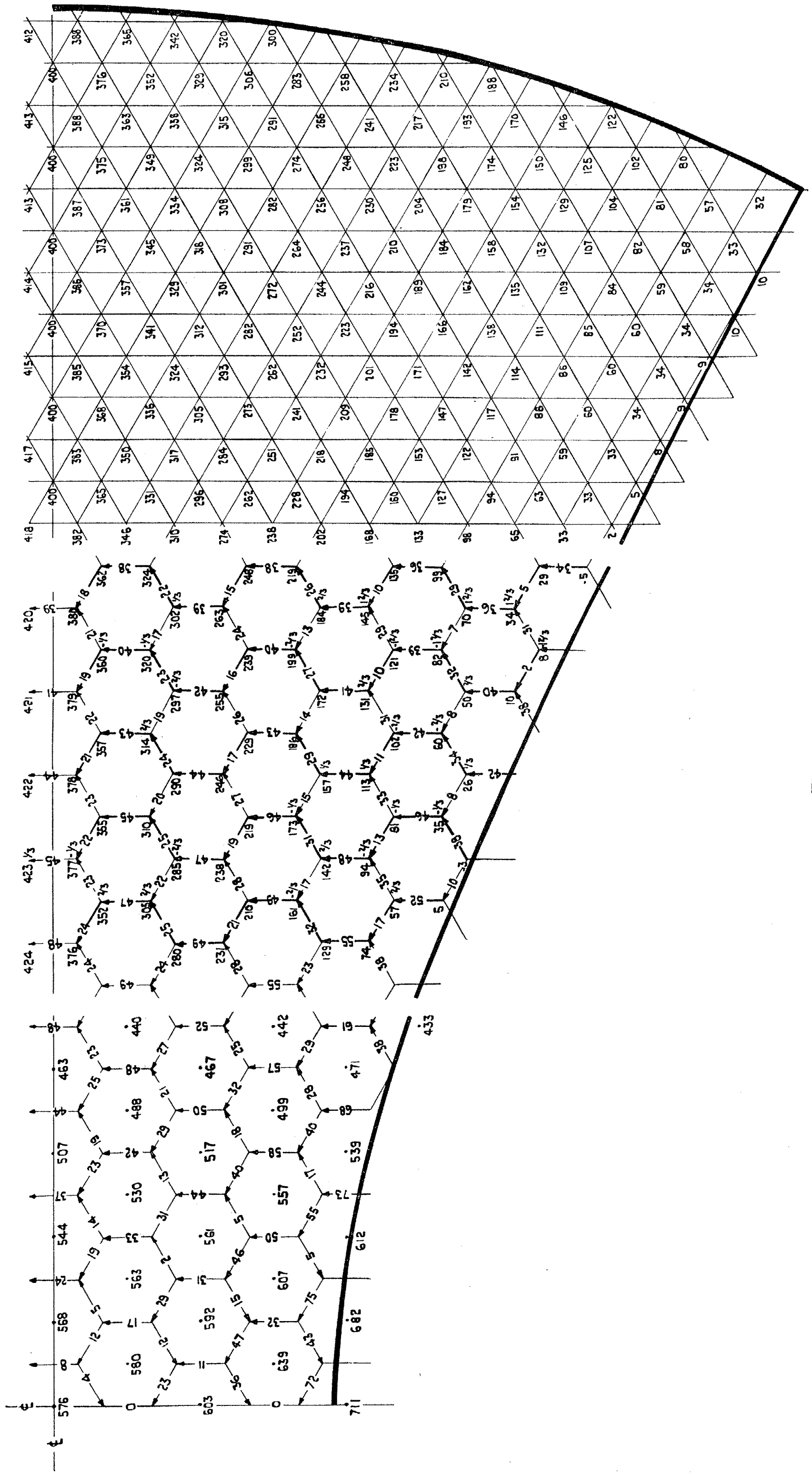

FIGURE 5

the $\Delta\phi$'s by summation along paths (arbitrarily chosen) which are indicated in bold lines.* These values, shown by figures to the left of the hexagon corners,† are in some instances incompatible with the $\Delta\phi$'s attached to the sides of hexagons which did not form paths of summation (and on that account are shown in fine lines): therefore residual forces are entailed in accordance with (13), and these are shown by figures to the right of the hexagon corners. Only a few had magnitudes in excess of 2/3—a figure which was neglected in our computation of ψ (cf. figure 2): all were brought below that magnitude by a final liquidation in relation to a finer triangular net introduced in the manner of § 10. The right-hand part of figure 5 shows the accepted approximation to ϕ.

It remains to allow for the arbitrary addition of 400 to the values of ϕ (cf. footnote) and for the multiplier introduced in § 6; then, to deduce values of α and β in accordance with the expressions (6), and using these to construct the "α–β net" (§ 1). Figure 6 exhibits our final solution of this problem in conformal transformation.

Reviewing §§ 7–10 we remark that when ψ is correct, then ϕ as calculated from the $\Delta\phi$'s not only satisfies the finite-difference approximation (14) to Laplace's equation, but is also single-valued and so does not call for adjustment. On this account, when ϕ is wanted it appears that labour will be saved in the end by carrying the liquidation of ψ further than would normally be necessary. In adjusting ϕ it should be remembered that its gradient normal to the boundary is specified and therefore (strictly) may not be altered.

II. Transformation of an interior region into a rectangle

12. This transformation also was noticed in the paper cited previously (§ 3). To illustrate it we transform a typical rail section (figure 1 *b*) into a rectangle such that one pair of opposite sides corresponds with the top and bottom of the rail, the other pair with its curved sides. The points which are to become corners of the rectangle have been chosen arbitrarily.

Here the shape of the boundary to be transformed makes a square mesh ($N = 4$) more convenient, and the relative simplicity of its boundary condition suggests the desirability of calculating first that function (here denoted by β) which has to take a constant value along each curved side. On account of symmetry only one half of the section need be considered if β is made zero on the centre-line: at internal nodes it must satisfy the finite-difference relation (akin to (14), but with $N = 4$)

$$\beta_0 = \tfrac{1}{4}\Sigma_{a,\,N}(\beta). \tag{15}$$

The computation of β, as of ψ in § 6, is a standard problem already treated in Part III of this series. Moreover, β being known its conjugate α can be determined by methods

* The paths should be roughly orthogonal to the boundary, since $\partial\phi/\partial\nu$ is specified.

† It is known (from the symmetry of ψ) that ϕ has a constant value along both the lines BA, BC in figure 2, also variations skew-symmetrical with respect to both. Accordingly the natural procedure would be to make the constant value zero; but in our calculations an arbitrary value 400 was taken with the aim of keeping ϕ positive.

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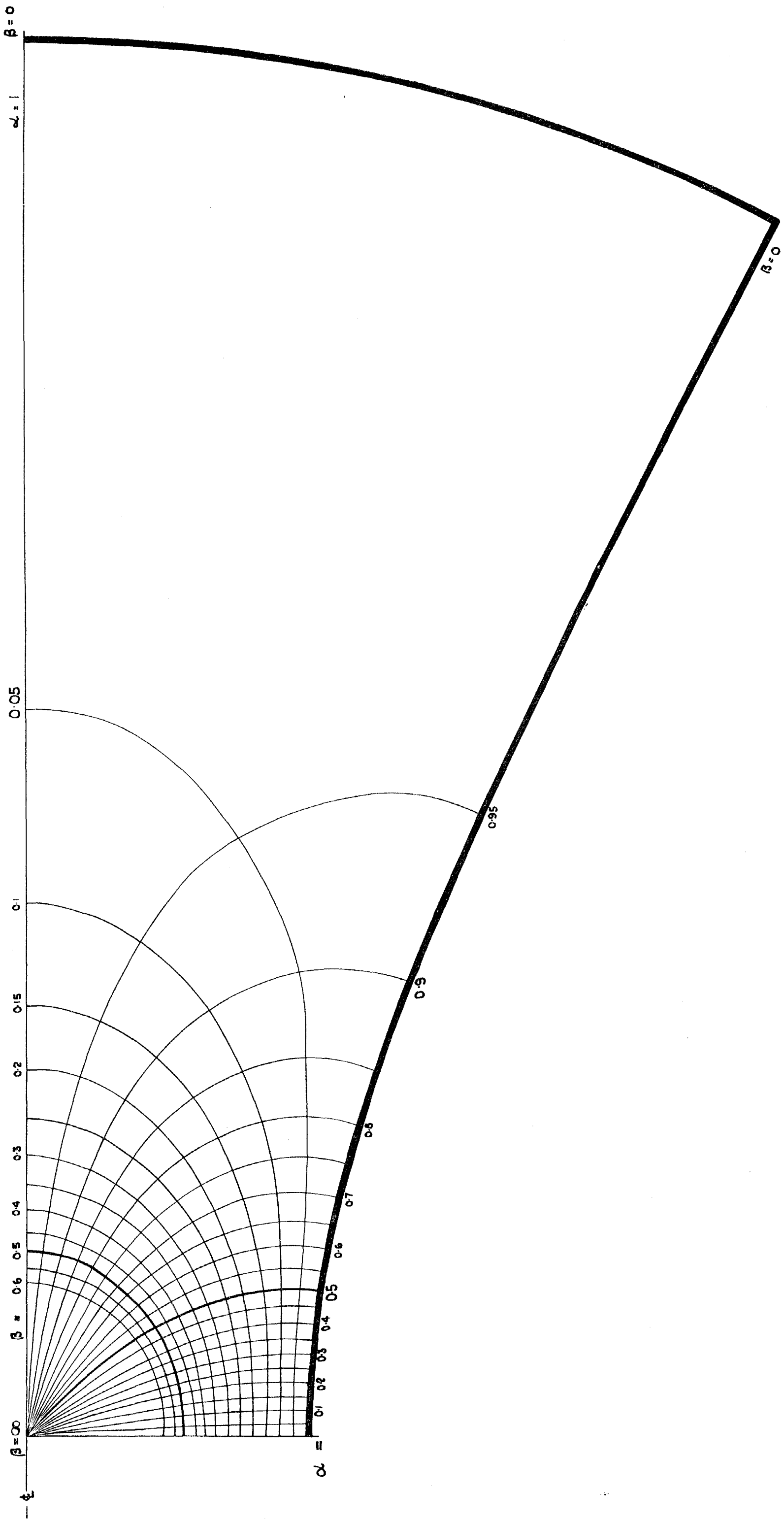

FIGURE 6

exactly similar to those of §§ 7–10, except that on account of the geometry of the square-mesh net the factor $\sqrt{3}$ in (11) is replaced by unity and N in (13) $= 4$. Therefore in regard to the actual transformation all that need be given is the accepted values of α and β (figure 7), together with the resulting "map".*

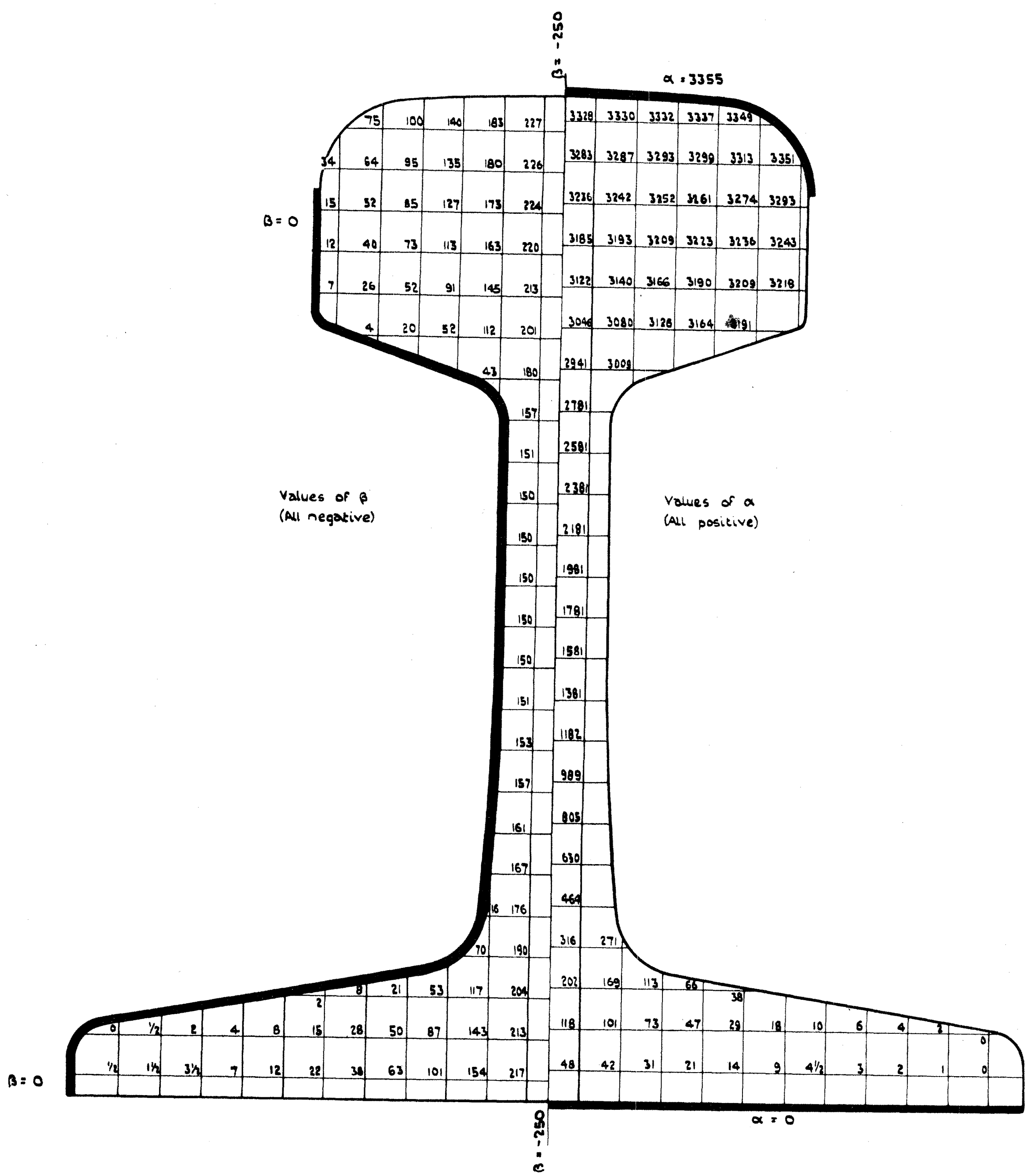

FIGURE 7

* The actual computations were carried to a net twice as fine as that shown in figure 7.

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13. The bold-line curves superposed on the map (figure 8) call on the other hand for explanation. They are contours of the quantity h defined in (2) of § 1, and they have been calculated from the expression

$$h^2 = \left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2 = \tfrac{1}{2}\nabla^2(\beta^2) - \beta\nabla^2\beta,$$

$$\approx \frac{2}{a^2}\left[\beta_0^2 + \frac{1}{N}\{\Sigma_{a,N}(\beta^2) - 2\beta_0\Sigma_{a,N}(\beta)\}\right], \tag{16}$$

when we substitute for ∇^2 its finite-difference approximation as presented in equation (9) of Part III. Rearranging the right-hand side of (16), we have

$$h^2 \approx \frac{2}{Na^2}\Sigma_{a,N}(\beta-\beta_0)^2 = \frac{2}{Na^2}\Sigma_{a,N}(\varDelta\beta^2), \tag{17}$$

in the notation of §§ 8 and 9, a standing (as before) for the distance between nodal points. This formula is very convenient in practice, used in conjunction with a table of squares and square roots. Some "smoothing" of results may be necessary (it will be realized that we are in effect performing a double differentiation of the computed β), and in this connexion (when high accuracy is wanted) use may be made of the fact that $\log h$, like β, is a plane-harmonic function of x and y.*

Geometrically interpreted, h is the ratio in which the sides of an infinitesimal rectangle are altered when the curvilinear is transformed into a rectilinear map as indicated in figure 1. The scale of the rectilinear map can of course be chosen arbitrarily: in figure 8, numbers attached to the bold-line contours give values of h on the assumption that $h = 1$ in the region of the "web" of the rail section, where the contours of α and β are approximately straight. Actually, of course, h must vary in this region; but its variation is too slight to be calculated accurately, and for that reason the contour $h = 1$, as being somewhat uncertain, is indicated by a broken line. Like uncertainties exist in regard to contours near the corners of the rail section, and these have been indicated similarly.

III. Transformation of an exterior region into the region exterior to a circle

14. This transformation is another special case of the problem stated in § 4: the shape of the inner boundary B is specified, and the outer boundary A is at infinity. We have to "map" the region external to B by contours of two conjugate plane-harmonic functions α, β, of which $\beta = 0$ at all points on B and at infinity tends to identity with $\log r/c$ (c being constant), while α is a cyclic function tending at infinity to identity with $-\theta$.

The special feature of the problem from the standpoint of approximate computation is

* Cf. § 17.

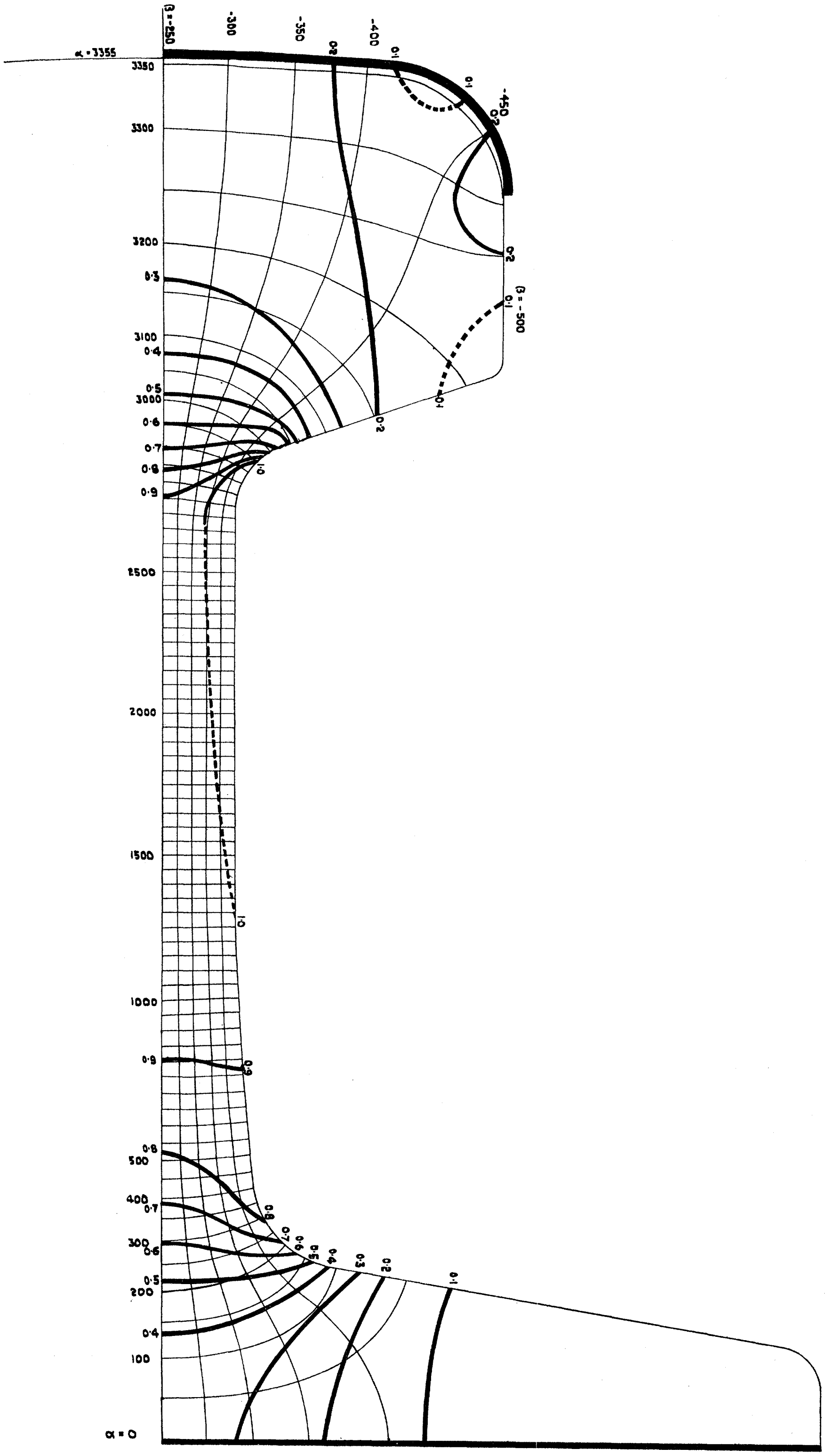

FIGURE 8

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the extent of the field within which β is to be determined. To circumvent it we now transform the problem by geometrical inversion, using

$$x' = \frac{c^2x}{x^2+y^2}, \quad y' = \frac{c^2y}{x^2+y^2} \tag{i}$$

as new co-ordinates in place of x and y. Then

$$r' = (x'^2+y'^2)^{\frac{1}{2}} = c^2/r, \quad \theta' = \tan^{-1}(y'/x') = \theta, \tag{ii}$$

and it is easy to prove that

$$\nabla'^2 \equiv \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2} \equiv \frac{r^4}{c^4}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right], \tag{iii}$$

so that any plane-harmonic function of x and y will also be plane-harmonic in x' and y'. The boundary B will invert into some new curve C (say), and the region external to B will invert into the region internal to C, points at infinity (A) inverting into the origin of co-ordinates.

15. Thus transformed the problem becomes exactly similar to the problem treated in §§ 4–10, except that $\alpha+i\beta$, being a function of $x+iy$ or $re^{i\theta}$, is now required to be a function of $r'e^{-i\theta'}$ or of $x'-iy'$, i.e. $\alpha-i\beta$ is required to be a function of $x'+iy'$.* The condition

$$\beta \to \log(r/c) \text{ as } r\to\infty$$

becomes

$$\beta \to \log(c/r') \text{ as } r'\to 0, \tag{iv}$$

by (ii) of § 14; and since β must vanish on B, in the transformed problem

$$\beta = 0 \text{ at all points on } C. \tag{v}$$

Proceeding on the lines of § 5, we write

$$-\alpha = \theta+\phi, \quad \beta = \log(c/r')+\psi, \tag{18}$$

thus requiring ψ to be plane-harmonic and without singularity at the origin, and $\phi+i\psi$ to be a function of $x'+iy'$. Then (v) imposes the relation

$$\psi = \log(r'/c) \text{ at points on } C, \tag{19}$$

and it remains to evaluate ψ and ϕ (approximately) at nodal points inside C. Contours of α and β can then be drawn in accordance with (18), and finally the required contours (external to B) can be obtained from these by another inversion.

16. As an example we now consider a typical airscrew section which was used by Bryant and Williams (1925) for a study of two-dimensional flow round an aerofoil in a wind-tunnel. Figure 9 shows the aerofoil section B, the chosen circle of inversion, and

* In order that $\alpha+i\beta$ may be a function of $x+iy$, the operation of inversion must be accompanied by "reflexion" with respect to the axis of x.

the resulting boundary C. Values of ψ on this "inverted boundary" were calculated from the formula

$$\psi = 1000 \log_{10}(r'/c), \tag{20}$$

and comparison of (20) with (19) shows that in consequence the calculated values of ψ contain a multiplying factor

$$1000 \log_{10} e = 434{\cdot}2945$$

(cf. § 6). Thus when ψ has the calculated values, and if values of ϕ (such that $\phi + i\psi$ is a function of $x+iy$) are deduced in the manner of §§ 7–11, the formulae (18) are replaced by

$$\left.\begin{aligned} -434{\cdot}2945\alpha &= 434{\cdot}2945\,\theta + \phi, \\ 434{\cdot}2945\beta &= 434{\cdot}2945 \log(c/r') + \psi, \\ &= 1000 \log_{10}(c/r') + \psi. \end{aligned}\right\} \tag{21}$$

ϕ being non-cyclic, $-\alpha$ like θ is cyclic with constant 2π, and contours of α, for equal spacing, must be drawn for submultiples of this quantity. We decided to plot 40 α-contours, i.e. for values differing by $\pi/20$: this means, according to (21),

$$\text{contours of } \frac{200}{\pi}\left(\theta + \frac{\phi}{434{\cdot}2945}\right) \text{ for values differing by 10,}$$

i.e.

$$\text{contours of } \frac{10}{9}\theta^\circ + \frac{200\,\phi}{\pi \times 434{\cdot}2945} \text{ for values differing by 10,}$$

where θ° stands for *θ expressed in angular degrees*. The contour values of β should be separated by the same interval as those of α: therefore according to (21) contours should be drawn for values of $\{\psi - 1000 \log_{10}(r'/c)\}$ differing by $434{\cdot}2945 \times \frac{\pi}{20}$, or by $68{\cdot}22$, and starting with C as the contour ($\beta = 0$).

Figure 9 shows the accepted values of ψ,* contours of β as deduced from these, and the contours which result by inversion. Figure 10 shows, similarly, the accepted values of ϕ, contours of α as deduced from these, and the contours which result from inversion. Near the "waist" of the "inverted boundary" C (corresponding with the leading and trailing edges of the aerofoil section) ϕ and ψ vary rapidly and contours are not easy to determine with accuracy. Accordingly in that region values were calculated for nodal points of a considerably finer net, (14) and the corresponding equation in ψ being used as formulae of interpolation. The finer net was extended into regions where the variation of ϕ or ψ was sufficiently slow to permit reliable estimation of boundary values: interpolation then becomes a standard problem in potential theory, treated already in Part III (Christopherson and Southwell 1938).

Combining the contours of figures 9 and 10 we have the required "α–β map" (§ 14).

* To avoid confusion, no values have been inserted within the circle of inversion.

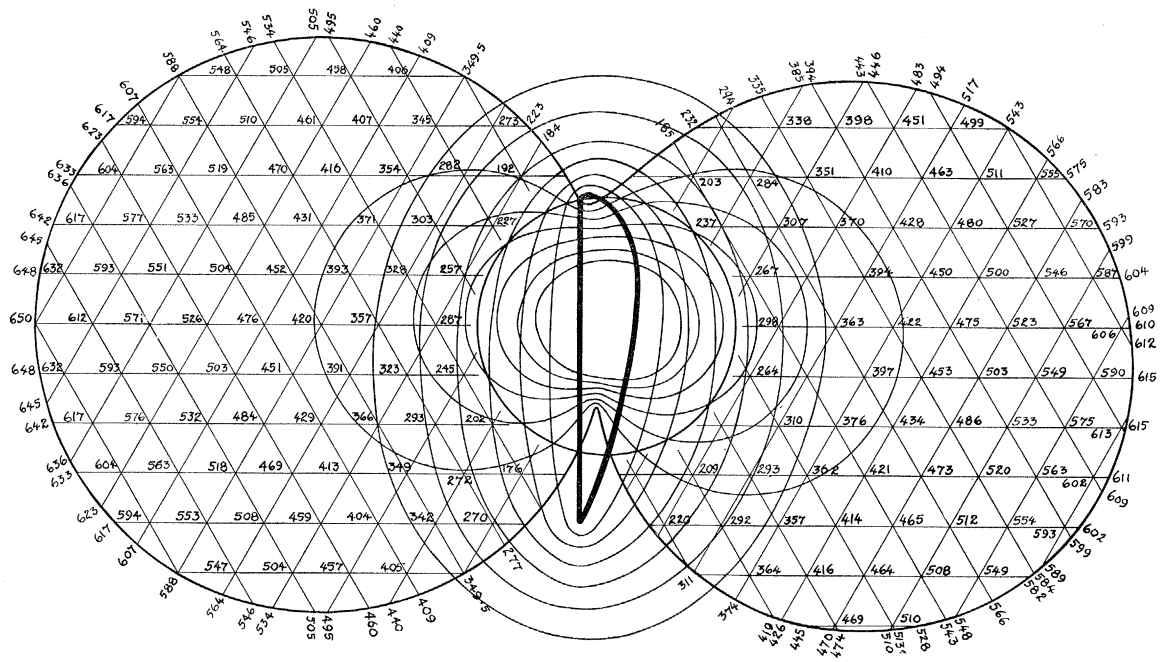

FIGURE 9

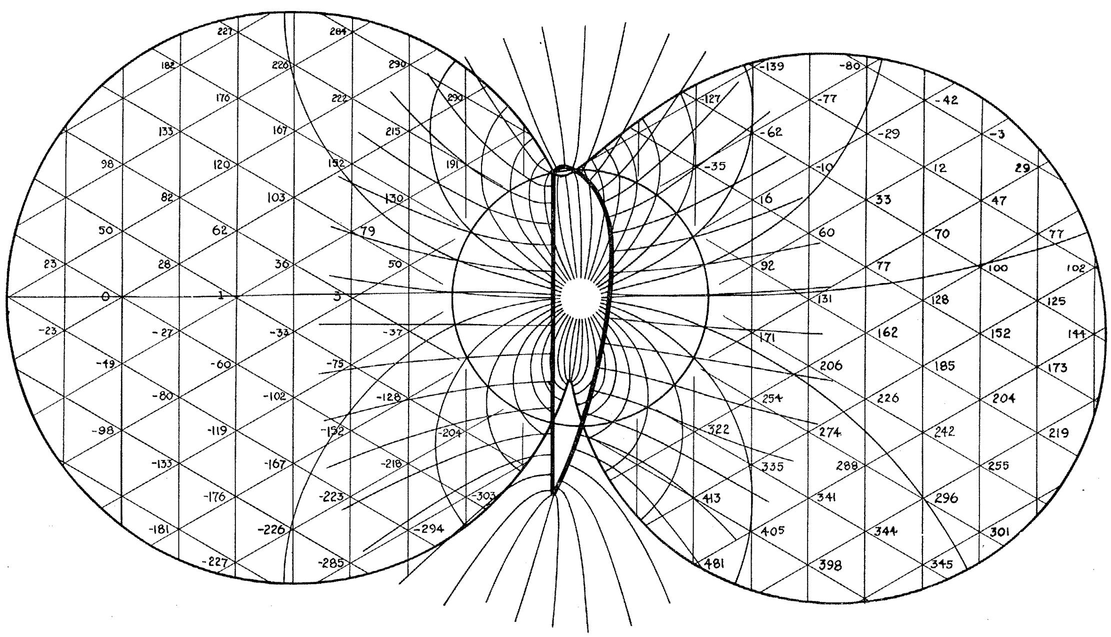

FIGURE 10

IV. TRANSFORMATION OF A SEMI-INFINITE REGION INTO AN INFINITE STRIP

17. Given a transformation of the type discussed in section III, we have only to apply the transformation

$$\phi + i\psi = k\left(z + \frac{a^2}{z}\right) \quad (z = x + iy) \tag{22}$$

in order that the circle $x^2+y^2 = a^2$ together with the line $y = 0$ may correspond with the axis $\psi = 0$, and the semi-infinite regions (external to the circle) in which $y \gtrless 0$ with the half-planes in which $\psi \gtrless 0$. It is thus unnecessary to consider further the problem of transformation into a half-plane, except in relation to *multiply-connected* regions which have symmetry such that in effect a region finite in one direction has to be transformed into a strip of infinite length and uniform breadth.

Figure 11 presents a problem of this kind which Poritsky and others have discussed in a recent paper (1939). Having determined the "α–β map" as shown, we can transform the notched strip into a strip with parallel sides; and then by mere repetition we can transform a plane containing one straight row of circular holes into a plane containing one straight row of parallel slits (cf. figure 1*d*). For the computations embodied in figure 11 we are indebted to Mr L. Fox and Miss A. Pellew.

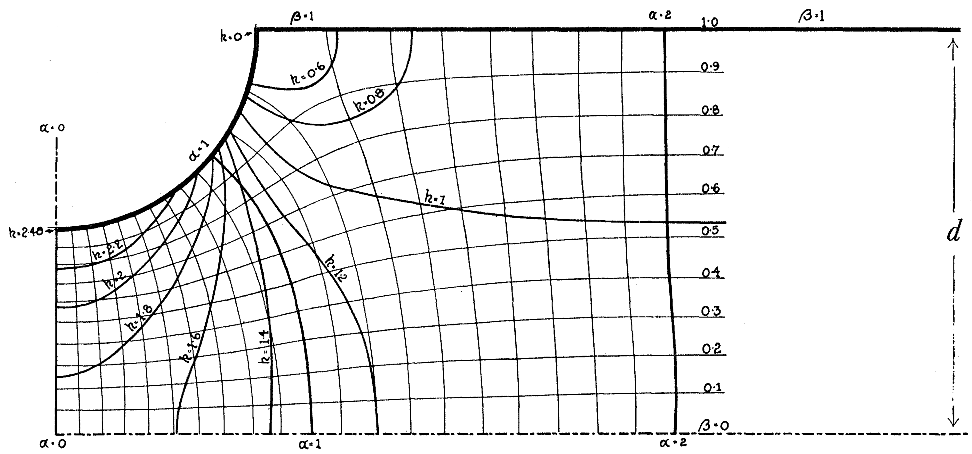

FIGURE 11

The solution was expressed in two parts by writing

$$\alpha = \frac{x}{d} - \phi, \quad \beta = \frac{y}{d} - \psi, \tag{23}$$

where x and y relate to an origin at the centre of the strip and d is the half-width of the unnotched part; and the methods of preceding sections were employed to find forms for ϕ and ψ which give β the value ± 1 on the two edges of the strip, i.e. a zero value on the centre line. On account of symmetry, only one-quarter of the field needs to be reproduced. Contours of α, β and h are plotted in figure 11.

As in § 13 the formula (17) was used to compute values of h, but in this instance the results were "smoothed" not as for figure 8 by judgement applied to cross-plottings but (as suggested in § 13) by taking advantage of the circumstance that $\log h$ is plane-harmonic. For each computed value of h the corresponding value of $\log h$ was recorded as a "displacement" of the appropriate nodal point; then the corresponding "residual forces", calculated from a formula of the type (13), were "liquidated" by the standard

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procedure of Part III. Liquidation was carried far enough to make the fractional uncertainty the same for h as for α and β, namely 0·1 %. In problems which call for an accurate determination of h this systematic procedure would appear to have advantages over the customary graphical treatment.

V. An electrical application

18. As a last example we consider conformal transformation into a circular annulus of the region contained within two closed boundaries. The problem has an electrical application (cf. § 3), since its solution gives in effect the electric capacity of a straight cable or condenser. We take the case of two square boundaries (figure 13).

If (as in § 4) β is zero on the outer boundary A and has the value 1 at every point of the inner boundary B, then β may be interpreted as electric potential and α as the current function (Bradfield *et al.* 1937): α will be cyclic with cyclic constant ρ/R, ρ being the (uniform) resistivity and R the total resistance of the conducting material between A and B. It can moreover be shown (Jeans 1923, § 386) that $4\pi R/\rho = 1/C$, where C is the capacity (per unit length)* of A and B regarded as electrodes of an air condenser. Hence we have

$$4\pi C = \frac{\rho}{R} = \oint \frac{\partial \alpha}{\partial s} ds = -\oint \frac{\partial \beta}{\partial \nu} ds \tag{24}$$

(ν denoting the outward normal), when the integral relates to any contour surrounding B and lying within the outer boundary A. (If B is accurately plane-harmonic, all such contours yield the same value for ρ/R).

Using (24) we can estimate C (or ρ/R) without difficulty, given values of β computed for some regular net. For such estimation the current function α is not required.

19. The net in figure 12 is of square mesh ($N=4$), but the procedure now to be described is applicable (*mutatis mutandis*) in cases where $N = 3$ or 6. Let ABC be part of a closed contour, A, B, C being centres of adjacent squares. The contribution of AB to the integral $-\oint \frac{\partial \beta}{\partial \nu} ds$ is given with a fractional error of order a^2 by $AB\frac{\beta_F-\beta_E}{EF}$, i.e. by $(\beta_F-\beta_E)$, and the contribution of BC to the same integral is given to a like approximation by $(\beta_F-\beta_G)$: dealing similarly with other portions of the contour, we have approximately, for the square contour of which ABC is part,

$$\begin{aligned} -\oint \frac{\partial \beta}{\partial \nu} ds &= (\beta_F+\beta_J+\dots)+(\beta_F+\beta_K+\dots)-(\beta_E+\beta_H+\dots)-(\beta_G+\beta_L+\dots), \\ &= [\Sigma(\beta) \text{ for nodes on the internal square} \dots JFK\dots] \\ &\quad -[\Sigma(\beta) \text{ for nodes on the external square} \dots HEQGL\dots] \end{aligned} \tag{25}$$

* So that C, like R/ρ, is "non-dimensional".

under the convention that the value at each corner (such as F) of the internal square is to be counted twice and the value at each corner (such as Q) of the external square is not to be counted. Similar expressions hold in relation to the other square contours which are indicated by broken lines in figure 12.

Computed values of β are given in that diagram; they are zero at all points on A and 256 at all points on B, so must be divided by 256 before insertion in (25). On this

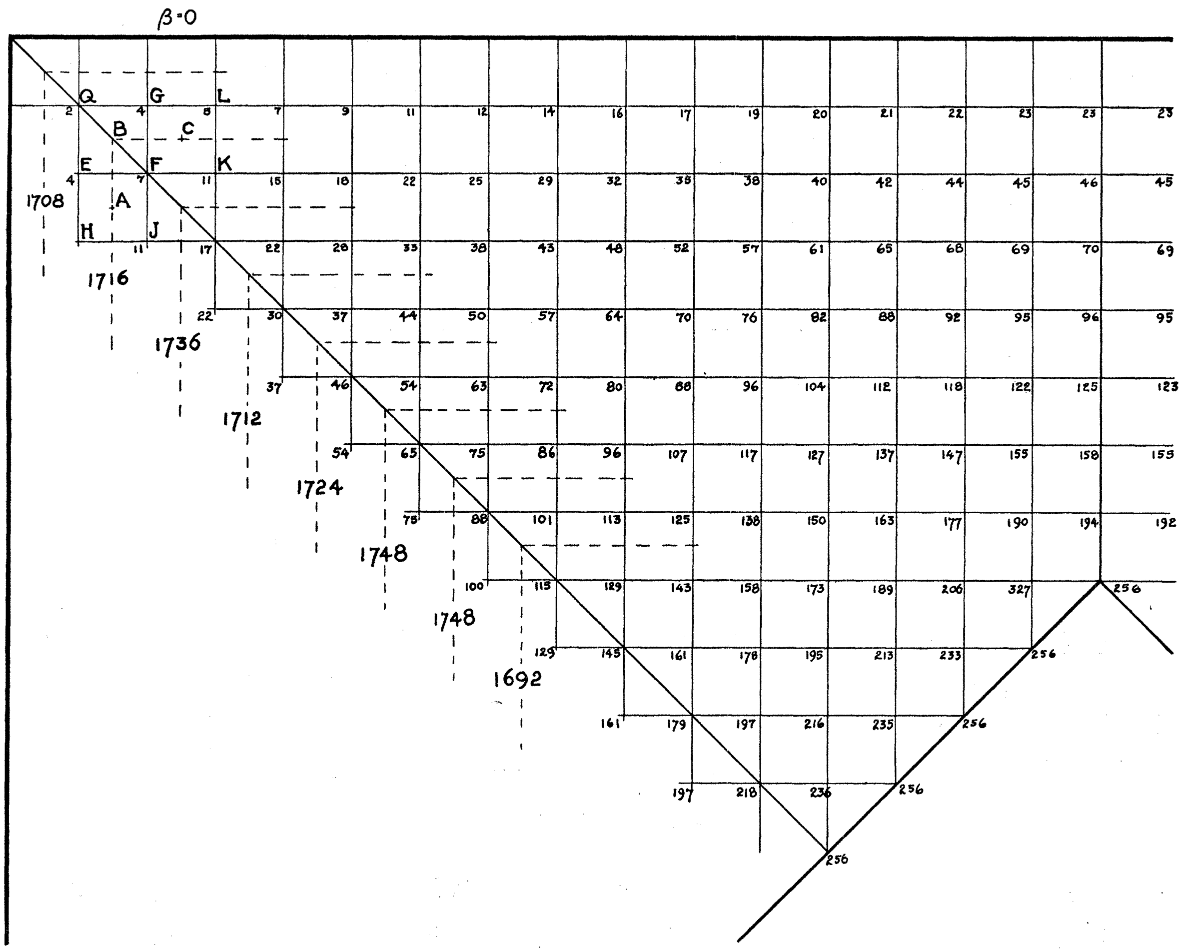

FIGURE 12

understanding, and taking account of symmetry, it is easy to verify that $-256\oint\frac{\partial\beta}{\partial\nu}ds$ takes for the different contours the values which are attached to them in figure 12. The accuracy of our approximations is likely to be least in relation to the inner contours, on account of the singularities at the corners of B: accordingly we reject the three innermost values, and take the mean (1719·2) of the other five. Then we have

$$-256\oint\frac{\partial\beta}{\partial\nu}ds = 1719{\cdot}2,$$

so that (24) becomes

$$4\pi C = \frac{\rho}{R} = \frac{1719{\cdot}2}{256} = 6{\cdot}71_5. \tag{26}$$

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The estimates obtained from the five individual contours differ from this mean estimate by amounts lying between $-11{\cdot}2$ and $16{\cdot}8$ in $1719{\cdot}2$, i.e. between $-0{\cdot}65$ and $0{\cdot}98$ %.

20. The problem just considered is the first (*a*) of the three cases which are shown diagrammatically in figure 13. Corresponding calculations were made for cases (*b*) and (*c*), and the resulting estimates were:

$$\left.\begin{aligned}&\text{In Case } (b)\text{: } 4\pi C = \frac{\rho}{R} = \frac{1683{\cdot}2}{256} = 6{\cdot}57_5 \text{ (estimate based on 5 outer contours)},\\ &\text{In Case } (c)\text{: } 4\pi C = \frac{\rho}{R} = \frac{1718{\cdot}9}{256} = 6{\cdot}71_5 \text{ (estimate based on 7 outer contours)}.\end{aligned}\right\} \quad (27)$$

We observe that the capacity fluctuates within very narrow limits as A is rotated relatively to B.

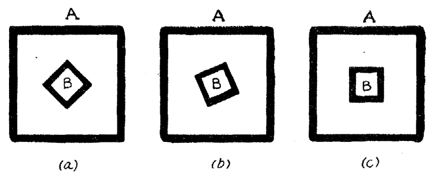

FIGURE 13

SUMMARY

Part III of this series dealt (*inter alia*) with the application of Relaxation Methods to problems in plane-potential theory. In this paper the problem of conformal transformation is discussed as a particular example.

Orthodox mathematics presents the transformation in an equation of the type

$$\alpha + i\beta = f(x+iy),$$

which expresses a one-to-one relation between points in the original and in the transformed region; but the equation is of no concern to the practical computer provided that he can construct "maps" of which it is the mathematical expression, and for this it is only necessary to have α and β evaluated at nodal points at some regular "net". Thus the problem from a practical standpoint is to construct the α–β contours. In this paper four cases of common occurrence are treated, and the "α–β maps" are reproduced. The accuracy of the calculations is in general greater than a drawing can exhibit.

Some incidental problems call for notice which were not confronted in Part III. Thus whereas one of the two conjugate functions α, β can be computed by the methods given previously, its computed values (being only an approximation to a plane-harmonic function) are not in general compatible with a single-valued conjugate. In Case (*c*) we are concerned with a region of infinite extent, and the device of "geometrical inversion" must be applied in order to render the problem tractable by relaxation

methods. For some applications we require values of the ratio $h = |d(\alpha+i\beta)/d(x+iy)|$ at least as accurate as those of α or β: examples are given to show that the requisite accuracy is attainable, use being made (when necessary) of the fact that $\log h$ is plane-harmonic.

The last problem treated (calculation of the electric capacity of a straight cable or condenser) is in essence an example of conformal transformation but can be solved when only one of α, β has been computed. It does not demand construction of the α–β map.

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